CUHK STAT1013: Data Science Toolbox Spring 2023

Statistics in Python III

Statistical simulation with application to Monty Hall Problem

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1 Monty Hall Problem

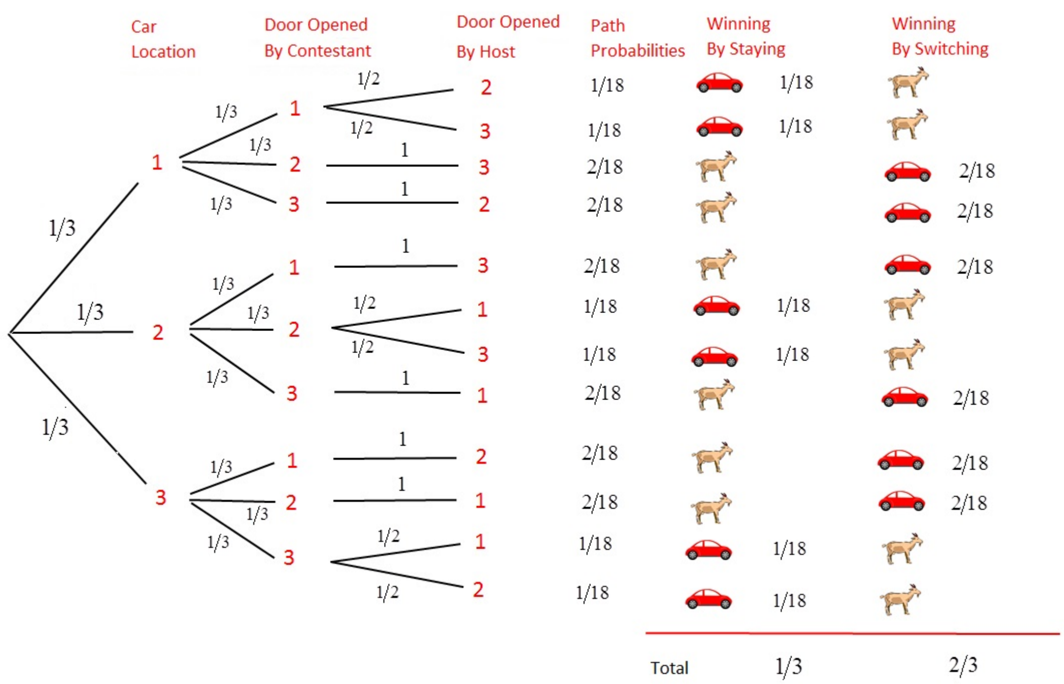
The Monty Hall problem is a brain teaser, in the form of a probability puzzle, loosely based on the American television game show *Let's Make a Deal* and named after its original host, *Monty Hall*.

- There are 3 doors, behind which are two goats and a car.
- You pick a door (call it door A). You're hoping for the car of course.
- Monty Hall, the game show host, examines the other doors (B & C) and opens one with a goat. (If both doors have goats, he picks randomly.)

2 Probabilistic solution

Denote $Y \in \{0,1\}$ is the outcome of the game, where $Y = 0$ indicates that you select a door with a goat, and $Y = 1$ indicate you select a door with a car.

2.1 Itemize all possibilities



2.2 Conditional probability

A statistical perspective: we want to make a decision D: "stay" or "switch", to maximize the probability, that is,

$$\max_{d=\text{"stay" or "switch"}} \mathbb{P}\left(Y = 1 \middle| D = d\right).$$

Thus, the primary goal is to compute the conditional probability. Clearly, $Y = 1$ is highly depended on your choice in the first stage, let's denote $X \in \{0, 1\}$ as your initial selection with or w/o car. Then, we have

$$\begin{aligned}\mathbb{P}\big(Y=1|D=d\big) &= \mathbb{P}\big(Y=1, X=0|D=d\big) + \mathbb{P}\big(Y=1, X=1|D=d\big) \\ &= \mathbb{P}\big(Y=1, |X=0, D=d\big)\mathbb{P}(X=0) + \mathbb{P}\big(Y=1|X=1, D=d\big)\mathbb{P}(X=1),\end{aligned}$$

where the first equality follows from the relation between marginal and joint probabilities, and the second equality follows Bayes' theorem. Note that

$$\begin{aligned}\mathbb{P}(X=1) &= \mathbb{P}(\text{initially select car}) = 1/3, \\ \mathbb{P}(X=0) &= \mathbb{P}(\text{initially select goat}) = 2/3.\end{aligned}$$

Now, we compare the difference in the conditional probabilities when making different decisions.

- $D =$ "stay"

$$\mathbb{P}\big(Y=1|D=\text{"stay"}\big) = \frac{2}{3}\mathbb{P}\big(Y=1|X=0, \text{"stay"}\big) + \frac{1}{3}\mathbb{P}\big(Y=1|X=1, \text{"stay"}\big) = \frac{1}{3}.$$

- $D =$ "switch"

$$\mathbb{P}\big(Y=1|D=\text{"switch"}\big) = \frac{2}{3}\mathbb{P}\big(Y=1|X=0, \text{"switch"}\big) + \frac{1}{3}\mathbb{P}\big(Y=1|X=1, \text{"switch"}\big) = \frac{2}{3}.$$

3 Understanding

Why Monty's actions help us? What's the result without Monty? Let's look at a variant Monty Hall problem.

- There are 100 doors to pick from in the beginning
- You pick one door
- Monty looks at the 99 others, finds the goats, and opens all but 1

Correct our misconceptions: Monty is taking a set of 99 choices and improving them by removing 98 goats. When he's done, he has the top door out of 99 for you to pick. Thus, Monty indeed provided some information!